*On Non-Homogeneous Equations with an Infinite Number of Variables.**

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1. *Introduction.*

In recent years important contributions to the problem of solving linear equations with an infinite number of variables have been made by Hill,† Poincaré,‡ von Koch,§ Hilbert,|| Toeplitz,¶ Schmidt,** and Bôcher and Brand.†† The most interesting and far-reaching developments of the theory which have been given up to the present time are those of the last two papers mentioned.

Two memoirs by Kötteritzsch‡‡ appeared earlier than any of those just referred to. They are formal in their treatment; that is to say, the requisite convergence proofs are not given, so that one is without information as to the range of validity of the results. These researches have usually been passed over in silence by those who have written on the subject of infinite systems of equations. To be sure they are incomplete in several respects and contain actual errors; but they do not deserve the neglect with which they have met.§§

The purpose of the present paper is to indicate an important range of validity for the results of Kötteritzsch. His formal solutions are derived directly and convergence proofs are supplied for two important classes of cases. The results are stated in a theorem at the close of the paper.

In connection with the results of § 5 I point out that *the methods of the present paper have a valuable range of applicability different from that of previously developed theories.*

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† *Acta Mathematicá*, VIII (1886), pp. 1–36. Previously published at Cambridge, U. S. A., in 1877.

‡ *Bulletin de la Société mathématique de France*, XIV (1886), pp. 77–90.

§ *Rendiconti del Circolo matematico di Palermo*, XXVIII (1909), pp. 255–266. See also the numerous references in this paper.

|| *Göttinger Nachrichten*, 1906, pp. 157–227; see especially pp. 218–227.

¶ *Rendiconti del Circolo matematico di Palermo*, XXVIII (1909), pp. 88–96. See also the references in this paper.

** *Rendiconti del Circolo matematico di Palermo*, XXV (1908), pp. 53–77.

†† *Annals of Mathematics*, XIII (1912), pp. 167–186.

‡‡ *Zeitschrift für Mathematik und Physik*, XV (1870), pp. 1–15, 229–268.

§§ Compare *Encyclopédie des Sciences Mathématiques*, I, pp. 319–321.

2. *Normal Form of the System.*

Let us consider the system of linear equations

$$\sum_{j=1}^{\infty} \bar{a}_{ij}\bar{u}_j=\bar{c}_i, \quad i=1, 2, \ldots, \tag{1}$$

having the property that no linear relation exists among any finite number of the first members. It is easy to see that the variables $\bar{u}_1, \bar{u}_2, \bar{u}_3, \ldots$ can be rearranged into a sequence $u_1, u_2, u_3, \ldots$ so that the system (1) can be reduced to an equivalent system of the form

$$\sum_{j=i}^{\infty} a_{ij}u_j=c_i, \quad a_{ii}=1, \quad i=1, 2, 3, \ldots, \tag{2}$$

the k-th equation in the new system being obtained by means of a linear combination of the first k equations of the old. We shall study the equations in the normal form (2). It is obvious that the results for this system can be carried over to system (1) without difficulty.

3. *The Formal Solution of Kötteritzsch.*

It is natural to expect that a solution of (2) is linear in the c's. Accordingly let us seek a solution of the form

$$u_k=\sum_{l=1}^{\infty} s_{kl}c_l, \quad k=1, 2, \ldots. \tag{3}$$

If we substitute this value of u_k from (3) into (2) we have formally

$$\sum_{j=i}^{\infty} a_{ij}\sum_{l=1}^{\infty} s_{jl}c_l=\sum_{l=1}^{\infty}\sum_{j=i}^{\infty} a_{ij}s_{jl}c_l=c_i, \quad i=1, 2, \ldots.$$

This is a formal identity in the c's provided that

$$\sum_{j=i}^{\infty} a_{ij}s_{jl}=\delta_{il}, \quad i, l=1, 2, 3, \ldots, \tag{4}$$

where δ_{il} is equal to 1 or zero according as i is or is not equal to l.

If we consider any fixed value of l, we have in (4) a singly infinite system of equations for determining those s's which have the given second subscript l. The l-th equation of the set has the second member 1; all the other equations of the set have the second member 0. It is obvious that there exists a particular solution $s_{kl}=\bar{s}_{kl}$ where $\bar{s}_{kl}=0$ when $k>l$ and $\bar{s}_{kl}=\Delta_{kl}$ when $k\leqq l$, Δ_{kl} being the determinant formed from

$$\Delta_l=\begin{vmatrix} 1 & a_{12} & a_{13} & a_{14} & \ldots & a_{1l} \\ 0 & 1 & a_{23} & a_{24} & \ldots & a_{2l} \\ 0 & 0 & 1 & a_{34} & \ldots & a_{3l} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & 0 & \ldots & 1 \end{vmatrix}$$

by replacing the last element in the k-th column by 1 and all the other elements in that column by 0. If now we let l vary over the range $1, 2, 3, \ldots,$ we obtain a particular solution of (4).

Consider now the homogeneous system corresponding to (2):

$$\sum_{j=i}^{\infty} a_{ij} v_j = 0, \quad i = 1, 2, \ldots. \tag{5}$$

This is also the homogeneous system corresponding to (4) for a fixed l and varying i. Consider the infinite set of solutions

$$v_j = v_{jk}, \quad k = 1, 2, \ldots,$$

of (5). These are assumed to be any solutions whatever of (5), whether the same or different; in particular we may have $v_{jk} = 0$ for all values of j and k.

Now, since $s_{kl} = \bar{s}_{kl}$ is a solution of (4), it is obvious that $s_{kl} = \bar{s}_{kl} + v_{kl}$ is also a solution of (4). This may be written in the form

$$s_{kl} = (k, l)\Delta_{kl} + v_{kl}, \quad k, l = 1, 2, \ldots,$$

where

$$(k, l) = \begin{cases} 0 \text{ when } k > l, \\ 1 \text{ when } k \leqq l. \end{cases}$$

Substituting these values of s_{kl} in (3) we have formally:

$$u_k = \sum_{l=1}^{\infty} \{(k, l)\Delta_{kl} + v_{kl}\} c_l, \quad k = 1, 2, \ldots. \tag{6}$$

If this value for u_k is substituted for u_k in (2), a formal identity in the c's will be obtained. Thus we have a formal solution of (2). If we put $v_{kl} = 0$, we have the simpler formal solution

$$u_k = \sum_{l=1}^{\infty} (k, l)\Delta_{kl} c_l = \sum_{l=k}^{\infty} \Delta_{kl} c_l, \quad k = 1, 2, \ldots. \tag{7}$$

This latter is essentially the formal solution of Kötteritzsch. It is the one which we shall study in the following sections of this paper.

In a previous paper* I have had occasion to solve a particular system of equations of the form (2). This example illustrates the method by which appropriate determinations of the quantities v_{kl} may be made, so as to ensure the convergence of (6) when the condition $v_{kl} = 0$ for every k and l gives rise to divergent series. The example also brings to notice the usefulness of this method when it is desired to select a particular solution (6) of (2) which has important properties in addition to satisfying the system of equations.

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* American Journal of Mathematics, XXXV (1913), pp. 164–175.

4. *Validity of the Formal Solution* (7) *as an Actual Solution.*

In the first place it is obvious that (7) cannot afford a solution of (2) unless every infinite series in (7) is convergent. But there are other conditions which must also be satisfied. Substituting from (7) into the first member of (2) we have

$$\sum_{j=i}^{\infty}\sum_{l=1}^{\infty} a_{ij}(j,\, l)\,\Delta_{jl}c_l. \tag{8}$$

Assuming for a moment the validity of an interchange in the order of summations in (8), we should have for the value of (8)

$$\sum_{j=i}^{\infty}\sum_{l=1}^{\infty} a_{ij}(j,\, l)\,\Delta_{jl}c_l=\sum_{l=1}^{\infty}\sum_{j=i}^{\infty} a_{ij}(j,\, l)\,\Delta_{jl}c_l=c_i,$$

since

$$\sum_{j=i}^{\infty} a_{ij}(j,\, l)\,\Delta_{jl}=\sum_{j=i}^{\infty} a_{ij}\bar{s}_{jl}=\delta_{il}.$$

Hence we conclude that (7) will afford a valid solution of (2) when and only when the series in (7) converges for every value of k and the interchange of the order of summations in (8) is legitimate.

Now we may think of (8) arranged as a double series:

$$\begin{array}{l} a_{ii}(i,\, 1)\,\Delta_{i1}c_1+a_{ii}(i,\, 2)\,\Delta_{i2}c_2+a_{ii}(i,\, 3)\,\Delta_{i3}c_3+\ldots\,, \\ a_{i,\, i+1}(i+1,\, 1)\,\Delta_{i+1,\, 1}c_1+a_{i,\, i+1}(i+1,\, 2)\,\Delta_{i+1,\, 2}c_2+a_{i,\, i+1}(i+1,\, 3)\,\Delta_{i+1,\, 3}c_3+\ldots\,, \\ a_{i,\, i+2}(i+2,\, 1)\,\Delta_{i+2,\, 1}c_1+a_{i,\, i+2}(i+2,\, 2)\,\Delta_{i+2,\, 2}c_2+a_{i,\, i+2}(i+2,\, 3)\,\Delta_{i+2,\, 3}c_3+\ldots\,, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{array}$$

The necessary convergence in (7) implies the convergence of every row of this double series. Then for the convergence of this double series it is necessary and sufficient* that the convergence of the rows be uniform and that the series

$$S_1+S_2+S_3+\ldots\,.$$

shall be convergent, where S_m is the sum of the m-th row of the double series.

Clearly the convergence of this double series is sufficient to ensure the legitimacy of the interchange in the order of summations in (8). Hence we conclude that (7) will be a valid solution of (2) in all cases when the convergence of the first series in (7) is uniform with respect to k and the series $\sum_{j=i}^{\infty} a_{ij}u_j$ is convergent, where u_j is the sum of the series (7) for $k=j$. We shall determine two important cases when these convergence conditions are satisfied.

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* See Hobson's *Theory of Functions of a Real Variable*, p. 466.

In developing each of these two conditions we shall have need of Hadamard's fundamental theorem* concerning an upper bound to the absolute value of a determinant. This theorem may be stated as follows:

If Δ is an n-th order determinant in which β_{ij} is the element in the i-th row and the j-th column, then

$$|\Delta| \leqq \sqrt{r_1 r_2 \ldots . r_n}, \qquad |\Delta| \leqq \sqrt{\rho_1 \rho_2 \ldots . \rho_n},$$

where

$$r_i = \sum_{j=1}^{n} |\beta_{ij}|^2, \qquad \rho_i = \sum_{j=1}^{n} |\beta_{ji}|^2.$$

5. *First Class of Cases.*

To obtain our first condition implying the validity of (7) as a solution of (2) we proceed thus: Denote by σ_i the sum

$$\sigma_i = \sum_{j=1}^{i} |a_{ji}|^2.$$

From the definition of Δ_{kl} and the second inequality for $|\Delta|$ in Hadamard's theorem we see that

$$|\Delta_{kl}|^2 \leqq \frac{1}{\sigma_k} \prod_{j=1}^{l} \sigma_j \leqq \sigma_1 \sigma_2 \ldots . \sigma_l .$$

From this it follows that the first series in (7) for u_k is term by term not greater in absolute value than the series

$$\sum_{l=1}^{\infty} \sqrt{\sigma_1 \sigma_2 \ldots . \sigma_l}\, |c_l|. \tag{9}$$

If this series converges, it follows that the first series for u_k is uniformly convergent with respect to k. Also, it is obvious that

$$|u_k| \leqq \sum_{l=k}^{\infty} \frac{\sqrt{\sigma_1 \sigma_2 \ldots . \sigma_l}}{\sqrt{\sigma_k}} |c_l| = A_k,$$

where A_k is defined by this relation. Consider the series

$$\sum_{j=i}^{\infty} |a_{ij}| A_j, \quad i=1, 2, \ldots . . \tag{10}$$

From our test in § 4 it follows that the convergence of series (9) and (10) is sufficient to ensure the validity of (7) as a solution of (2).

Since $|u_k|$ is evidently not greater than the sum of (9), it follows from our test in § 4 that the convergence of (9) and of the series

$$\sum_{j=i}^{\infty} |a_{ij}|, \quad i=1, 2, \ldots ., \tag{11}$$

is also sufficient for the validity of (7) as a solution of (2).

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* *Bulletin des Sciences Mathématiques* (Darboux), XVII (1893), pp. 240–246.

A comparison of these results with the theory of von Koch and also with that of Schmidt will be sufficient to show that *the methods of the present paper have an important range of applicability different from that of previously developed theories.* To take a fairly obvious case, let us consider a system of equations (2) in which

$$|a_{ij}| \leqq M^{i+j},$$

where M is independent of i and j. It is easy to see that the numbers c_1, c_2, c_3, ... can be chosen in an infinity of ways so as to ensure that the requisite convergence conditions are satisfied. A comparison of this case with those treated by von Koch and Schmidt is sufficient to justify the statement made above.

6. *Second Class of Cases.*

To obtain a second condition under which (7) affords an actual solution of (2) let us assume with Schmidt the convergence of the series $|a_{i1}|^2+|a_{i2}|^2+\ldots.$ for every value of i. We write

$$s_i=\sum_{j=1}^{\infty}|a_{ij}|^2, \quad i=1, 2, \ldots..$$

Applying the first inequality for $|\Delta|$ in Hadamard's theorem we see that

$$|\Delta_{kl}|^2 \leqq s_0 s_1 s_2 \ldots . s_{l-1}, \quad s_0=2,$$

since the sum of the squares of the absolute values of the elements in the i-th row of Δ_{kl}, $i \neq l$, is equal to or less than s_i, while the corresponding sum for the l-th row is 2. From this it follows that the first series in (7) for u_k is term by term not greater in absolute value than the series

$$\sum_{l=1}^{\infty}\sqrt{s_0 s_1 \ldots . s_{l-1}}\,|c_l|. \tag{12}$$

If this series converges, it follows that the first series for u_k is uniformly convergent with respect to k. Also, it is obvious that

$$|u_k| \leqq \sum_{l=k}^{\infty}\sqrt{s_0 s_1 \ldots . s_{l-1}}\,|c_l|=B_k,$$

where B_k is defined by this relation. Consider the series

$$\sum_{j=i}^{\infty}|a_{ij}|B_j, \quad i=1, 2, \ldots.. \tag{13}$$

From our test in § 4 it follows that the convergence of series (12) and (13) is sufficient to ensure the validity of (7) as a solution of (2).

Since B_k obviously decreases with increase of k, it follows from the above result that the convergence of (12) and (11) is also sufficient for the validity of (7) as a solution of (2).

7. *Statement of the Principal Results.*

The principal results which we have obtained may be stated as follows:

THEOREM. *If the system of equations*

$$\sum_{j=1}^{\infty} \bar{a}_{ij}\bar{u}_j=\bar{c}_i, \quad i=1, 2, \ldots, \tag{$\bar{\mathrm{A}}$}$$

has the property that no linear relation exists among any finite number of the first members, it can readily be reduced to an equivalent system

$$\sum_{j=i}^{\infty} a_{ij}u_j=c_i, \quad a_{ii}=1, \quad i=1, 2, \ldots. \tag{A}$$

Denote by Δ_{kl} *the determinant formed from*

$$\begin{vmatrix} 1 & a_{12} & a_{13} & a_{14} & \ldots & a_{1l} \\ 0 & 1 & a_{23} & a_{24} & \ldots & a_{2l} \\ 0 & 0 & 1 & a_{34} & \ldots & a_{3l} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \ldots & 1 \end{vmatrix}$$

by replacing the last element in the k-th column by 1 *and all the other elements in that column by zero. Denote by* v_{kl}, $k=1, 2, \ldots$, *any solution* $v_k=v_{kl}$ *of the system*

$$\sum_{j=i}^{\infty} a_{ij}v_j=0, \quad i=1, 2, \ldots.$$

Then a formal solution of equations (A) *is*

$$u_k=\sum_{l=1}^{\infty} \{(k, l)\Delta_{kl}+v_{kl}\}c_l, \quad k=1, 2, \ldots, \tag{B}$$

where (k, l) *is unity when* $l \geqq k$, *and is otherwise zero. If we take* $v_{kl}=0$, *this reduces to*

$$u_k=\sum_{l=k}^{\infty} \Delta_{kl}c_l, \quad k=1, 2, \ldots. \tag{C}$$

Among the cases in which (C) *certainly affords an actual solution of* (A), *are the two general classes in which the one or the other of the two following conditions are satisfied:*

1. *If*

$$\sigma_i=\sum_{j=1}^{i} |a_{ji}|^2,$$

then the series

$$\sum_{l=1}^{\infty} \sqrt{\sigma_1\sigma_2\ldots\sigma_l}\,|c_l|\,; \qquad \sum_{j=i}^{\infty} |a_{ij}|A_j, \quad i=1, 2, \ldots,$$

converge, where A_j *has either of the two values* $A_j=1$, $j=1, 2, \ldots$, *or*

$$A_j=\sum_{l=j}^{\infty}\frac{\sqrt{\sigma_1\sigma_2\ldots\sigma_l}}{\sqrt{\sigma_j}}|c_l|,\quad j=1, 2, \ldots.$$

2. *The series* $|a_{i1}|^2+|a_{i2}|^2+|a_{i3}|^2+\ldots$ *converges to the sum* s_i. *The series*

$$\sum_{l=1}^{\infty}\sqrt{s_0s_1\ldots s_{l-1}}|c_l|,\quad s_0=2;\qquad \sum_{j=i}^{\infty}|a_{ij}|B_j,\quad i=1, 2, \ldots,$$

converge, where B_j *has either one of the two values* $B_j=1$, $j=1, 2, \ldots$, *or*

$$B_j=\sum_{l=j}^{\infty}\sqrt{s_0s_1\ldots s_{l-1}}|c_l|,\quad j=1, 2, \ldots.$$

From a solution of (A) *a solution of* ($\bar{\mathrm{A}}$) *may readily be found.*

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